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ON TWO INITIAL BOUNDARY VALUE PROBLEMS FOR THE GENERALIZED KdV EQUATION

Results on nonlocal well–posedness are established for initial boundary value problems for the generalized KdV equation in a bounded rectangle and a half–strip under natural assumptions on initial and boundary data. The considerable part of the study is devoted to special solutions of the linearized KdV equation of the "boundary potential"type.

In this paper problems of nonlocal well–posedness of initial boundary value problems in a rectangle $Q_T = (0,T) \times (0,1)$ and a half-strip $\Pi_T^- = (0,T) \times \mathbb{R}_-$ ($\mathbb{R}_- = (-\infty, 0), T > 0$) are studied for the generalized Korteweg – de Vries (KdV) equation

$$u_t + u_{xxx} + g(u)u_x = f(t,x). \tag{1}$$

For the problems in these domains we set initial condition

$$u(0,x) = u_0(x) \tag{2}$$

and the following boundary conditions: for the problem in Q_T

$$u(t,0) = u_1(t), \qquad u(t,1) = u_2(t), \qquad u_x(t,1) = u_3(t), \tag{3}$$

and for the problem in Π_T^-

$$u(t,0) = u_2(t), \qquad u_x(t,0) = u_3(t). \tag{4}$$

The function g satisfies the following growth restriction condition:

$$|g'(u)| \le c_0 = \text{const} \qquad \forall u \in \mathbb{R}. \tag{5}$$

First results on nonlocal resolvability and well–posedness of such problems were established in [1] for the KdV equation itself

$$u_t + u_{xxx} + au_x + uu_x = f(t,x) \tag{6}$$

($a = \text{const} \in \mathbb{R}$) under zero boundary conditions (3) and (4), $a = 0$ and $f \equiv 0$. The present paper continues the papers [2]–[5], where these problems were studied for the equations (1) and (6) in the general case.

Now we introduce some notations. For $s \in \mathbb{R}$ let

$$H^s(\mathbb{R}) = \left\{ f : \mathcal{F}^{-1}\left[(1+|\xi|)^s \widehat{f}(\xi)\right] \in L_2(\mathbb{R}) \right\}$$

($\widehat{f} = \mathcal{F}[f]$ and $\mathcal{F}^{-1}[f]$ are the direct and inverse Fourier transforms respectively). For any interval $I \subset \mathbb{R}$ we denote by $H^s(I)$ a space of restrictions on I of functions from $H^s(\mathbb{R})$ with natural restriction norm. We also use for $s \in (0,1)$ and $p \in [1, +\infty)$ the Slobodetskii space

$$W_p^s(I) = \left\{ f \in L_p(I) : \iint\limits_{I\times I} \frac{|f(t) - f(\tau)|^p}{|t-\tau|^{1+sp}} dt d\tau < \infty \right\}.$$

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It is well-known, that $H^s(I) = W_2^s(I)$ (see, for example [6]). We denote by $C_b(\overline{I})$ a space of continuous bounded on $\overline{I}$ functions. If X is a Banach space, let $C_b(\overline{I}; X)$ ($C_{b,w}(\overline{I}; X)$) be spaces of bounded and continuous (respectively weakly continuous) mappings from $\overline{I}$ to X.

By the symbol $\eta(x)$ we denote a certain "cut-off"function, namely, $\eta \in C^\infty(\mathbb{R})$, $\eta \geq 0$, $\eta' \geq 0$, $\eta(x) = 0$ for $x < 1/4$, $\eta(x) = 1$ for $x > 3/4$, $\eta'(x) > 0$ for $x \in (1/4, 3/4)$, $\eta(x) + \eta(1-x) \equiv 1$.

The essential part of investigations of the papers [2]–[5] was carried out with the use of a function

$$J_0(t,x;\mu) \equiv \frac{3}{4}(1 + 3\mathrm{sign}x) \int_0^t \frac{1}{t-\tau} A''\Big(\frac{x}{(t-\tau)^{1/3}}\Big)\mu(\tau)\,d\tau$$

($t \geq 0$, $x \neq 0$), where

$$A(\theta) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\xi^3 + \theta\xi)}\,d\xi \equiv \mathcal{F}^{-1}[e^{i\xi^3}](\theta) \tag{7}$$

is the Airy function. The function J_0 was introduced for functions $\mu(t)$, defined for $t \geq 0$, and is a special solution of the linearized KdV equation

$$v_t + v_{xxx} = 0 \tag{8}$$

of the "boundary potential"type. For the first time this function was considered in the paper [7] for $x \in \mathbb{R}_+ = (0, +\infty)$. Certain properties of the potential J_0 were studied in [3]. With the use of these properties in the papers [4] and [5] some results on nonlocal well-posedness of the problems in Q_T and Π_T^- for the equation (1) in classes of generalized solutions were established for $u_0 \in L_2$, $u_1 \in (L_{6+\varepsilon} \cap W_1^{1/3} \cap W_2^{1/6})(0,T)$, $u_2 \in W_1^{5/6+\varepsilon}(0,T)$, $u_3 \in L_2(0,T)$.

However, these conditions on smoothness of boundary data u_1 and u_2 are non-optimal. Consider the initial value problem for the equation (8) with the initial condition

$$v(0,x) = u_0(x). \tag{9}$$

If $u_0 \in H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, then for a solution of this problem $v(t,x) \in C_b(\mathbb{R}^t; H^s(\mathbb{R}^x))$ for any $x \in \mathbb{R}$

$$\|v(\cdot,x)\|_{\dot{H}^{(s+1)/3}(\mathbb{R}^t)} = \||\lambda|^{(s+1)/3}\widehat{v}(\lambda,x)\|_{L_2(\mathbb{R}^\lambda)} = c_0(s)\|u_0\|_{\dot{H}^s(\mathbb{R})},$$
$$\|v_x(\cdot,x)\|_{\dot{H}^{s/3}(\mathbb{R}^t)} = c_1(s)\|u_0\|_{\dot{H}^s(\mathbb{R})}$$

(see, for example, [8]). Therefore, one can assume, that conditions of the type $u_0 \in H^s$, $u_1, u_2 \in H^{(s+1)/3}(0,T)$, $u_3 \in H^{s/3}(0,T)$ are natural for the problems considered.

One of the main ideas of this paper in comparison with [2]–[5] is the substitution of the function J_0 by another solution of the equation (8), which is introduced for functions μ defined on the whole real line. This approach made it possible to establish results on nonlocal well-posedness of the considered problems under natural (or close to natural) assumptions on boundary data.

DEFINITION 1. *For any $t \in \mathbb{R}$ and any $x > 0$ let*

$$J(t,x;\mu) \equiv 3 \int_{-\infty}^t \frac{1}{t-\tau} A''\Big(\frac{x}{(t-\tau)^{1/3}}\Big)\mu(\tau)\,d\tau, \tag{10}$$

where the function A is defined by the formula (7).

LEMMA 1. *Let $\mu \in L_p(\mathbb{R})$ for some $p \in [1, +\infty]$. Then the function J is defined and infinitely differentiable for $t \in \mathbb{R}$, $x > 0$, satisfies the equation (8) in this domain and for any $x_0 > 0$ and non-negative integers m and l*

$$\sup_{x \geq x_0, t \in \mathbb{R}} |D_t^m D_x^l J(t, x; \mu)| \leq c(m, l, x_0, p) \|\mu\|_{L_p(\mathbb{R})}. \tag{11}$$

Proof. It is well-known, that the Airy function A is infinitely differentiable on $\mathbb{R}$, for any θ satisfies the equation

$$A''(\theta) = \frac{1}{3}\theta A(\theta) \tag{12}$$

and decreases rapidly with all its derivatives while $\theta \to +\infty$. Therefore, for $x \geq x_0 > 0$

$$|D_x^l J| \leq \int_{-\infty}^{t} \tfrac{3}{(t-\tau)^{1+l/3}} \left| A^{(l+2)}\Big(\tfrac{x}{(t-\tau)^{1/3}}\Big) \mu(\tau) \right| d\tau \leq$$
$$\leq 3^{1/q} x^{3/q-3-l} \left\| \theta^{3+l-4/q} (\theta A(\theta))^{(l)} \right\|_{L_q(\mathbb{R}_+)} \|\mu\|_{L_p(\mathbb{R})} \leq c(l, x_0, p) \|\mu\|_{L_p(\mathbb{R})},$$

where $1/p + 1/q = 1$. Besides that, it follows from (12), that $J_t + J_{xxx} = 0$. The lemma is proved.

LEMMA 2. *Let $\mu \in W_1^{1/3}(\mathbb{R})$. Then $J_x \in L_1(\mathbb{R}^t; C_b(\overline{\mathbb{R}}_+^x))$ and*

$$\|J_x(\cdot, \cdot; \mu)\|_{L_1(\mathbb{R}^t; C_b(\overline{\mathbb{R}}_+^x))} \leq c \|\mu\|_{W_1^{1/3}(\mathbb{R})}. \tag{13}$$

Proof. According to (10) for $x > 0$

$$J_x = \int_{-\infty}^{t} \tfrac{3}{(t-\tau)^{4/3}} A'''\Big(\tfrac{x}{(t-\tau)^{1/3}}\Big) \mu(\tau)\, d\tau =$$
$$= \int_{-\infty}^{t} \tfrac{3}{(t-\tau)^{4/3}} A'''\Big(\tfrac{x}{(t-\tau)^{1/3}}\Big) \big(\mu(\tau) - \mu(t)\big)\, d\tau,$$

because in virtue of (12)

$$\frac{1}{(t-\tau)^{4/3}} A'''\Big(\frac{x}{(t-\tau)^{1/3}}\Big) = \frac{\partial}{\partial \tau} \left[\frac{1}{(t-\tau)^{1/3}} A\Big(\frac{x}{(t-\tau)^{1/3}}\Big) \right].$$

Therefore, since $|A'''(\theta)| \leq \text{const}$ for $\theta \geq 0$, we find, that

$$\int_{\mathbb{R}} \sup_{x>0} |J_x(t, x; \mu)|\, dt \leq c \int_{\mathbb{R}} \int_{-\infty}^{t} \frac{|\mu(t) - \mu(\tau)|}{(t-\tau)^{4/3}}\, d\tau dt \leq c \|\mu\|_{W_1^{1/3}(\mathbb{R})}.$$

The proof is completed.

Now we obtain an alternative representation for the boundary potential J.

LEMMA 3. *Let $\mu \in L_p(\mathbb{R})$ for some $p \in [1,2]$. Then for any $x > 0$*

$$J(t,x;\mu) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{it\lambda} e^{-\frac{x}{2}(\sqrt{3}|\lambda|^{1/3} + i\lambda^{1/3})} \widehat{\mu}(\lambda)\, d\lambda \equiv \mathcal{F}_t^{-1}\left[e^{r(\lambda)x}\widehat{\mu}(\lambda)\right](t), \tag{14}$$

where

$$r(\lambda) = -\frac{1}{2}(\sqrt{3}|\lambda|^{1/3} + i\lambda^{1/3}). \tag{15}$$

Proof. At first let $\mu \in C_0^\infty(\mathbb{R})$. We put

$$G(t,x) \equiv \frac{1}{t^{1/3}} A\left(\frac{x}{t^{1/3}}\right) = \frac{1}{3}\mathcal{F}_t^{-1}\left[\lambda^{-2/3} e^{ix\lambda^{1/3}}\right](t).$$

Then

$$J(t,x;\mu) = D_x^2\big[3\big(G(\cdot,x)\vartheta(\cdot) * \mu\big)(t)\big], \tag{16}$$

where ϑ is the Heaviside function. For any $x > 0$ we derive, that

$$\mathcal{F}_t\big[3\big(G(\cdot,x)\vartheta(\cdot) * \mu\big)\big](\lambda) = \tfrac{1}{2\pi}\left(\lambda^{-2/3} e^{ix\lambda^{1/3}} * \mathcal{F}[\vartheta]\right)(\lambda)\widehat{\mu}(\lambda) =$$

$$= \left(\tfrac{1}{2}\lambda^{-2/3} e^{ix\lambda^{1/3}} + \tfrac{i}{2\pi}\text{v.p.}\int\limits_{\mathbb{R}} \tfrac{\xi^{-2/3} e^{ix\xi^{1/3}}}{\xi - \lambda}\, d\xi\right)\widehat{\mu}(\lambda) =$$

$$= \tfrac{1}{2}\lambda^{-2/3} e^{-\frac{x}{2}(\sqrt{3}|\lambda|^{1/3} + i\lambda^{1/3})}(1 - i\sqrt{3}\text{sign }\lambda)\widehat{\mu}(\lambda).$$

Substituting the obtained equality into (16) we derive (14). In the general case one can establish this equality with the use of closure.

LEMMA 4. *Let $\mu \in H^{(s+1)/3}(\mathbb{R})$ for some $s \geq 0$. Then $J \in C_b(\mathbb{R}^t; H^s(\mathbb{R}_+))$ and for every $t \in \mathbb{R}$*

$$\|J(t,\cdot;\mu)\|_{H^s(\mathbb{R}_+)} \leq c(s)\|\mu\|_{H^{(s+1)/3}(\mathbb{R})}. \tag{17}$$

Proof. Let, at first, $s = n$, where n is a non–negative integer number. According to (14) for $x > 0$

$$D_x^n J = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{it\lambda} r^n(\lambda) e^{r(\lambda)x}\widehat{\mu}(\lambda)\, d\lambda = \frac{3}{2\pi}\int\limits_{\mathbb{R}} e^{it\xi^3} r^n(\xi^3) e^{r(\xi^3)x}\widehat{\mu}(\xi^3)\xi^2\, d\xi.$$

We use the following inequality from [9]: if certain continuous function $\gamma(\lambda)$ satisfies an inequality $\Re\gamma(\lambda) \leq -\varepsilon|\lambda|$ for some $\varepsilon > 0$ and all $\lambda \in \mathbb{R}$, then

$$\Big\|\int\limits_{\mathbb{R}} e^{\gamma(\lambda)x} f(\lambda)\, d\lambda\Big\|_{L_2^x(\mathbb{R}_+)} \leq c(\varepsilon)\|f\|_{L_2(\mathbb{R})}.$$

Then since $\Re r(\xi^3) = -\sqrt{3}|\xi|/2$

$$\|D_x^n J\|_{L_2(\mathbb{R}_+)} \leq c\|\xi^{n+2}\widehat{\mu}(\xi^3)\|_{L_2(\mathbb{R})} \leq c_1\|\mu\|_{H^{(n+1)/3}(\mathbb{R})}.$$

Using interpolation in the spaces H^s we complete the proof.

LEMMA 5. *Let $\mu \in H^{n/3+s}(\mathbb{R})$ for some $s \geq 0$ and integer $n \geq 0$. Then $D_x^n J \in C_b(\overline{\mathbb{R}}_+^x; H^s(\mathbb{R}^t))$ and for every $x > 0$*

$$\|D_x^n J(\cdot, x; \mu)\|_{H^s(\mathbb{R})} \leq c(n)\|\mu\|_{H^{n/3+s}(\mathbb{R})}. \tag{18}$$

In addition,

$$D_x^n J(t, 0+0; \mu) = \mathcal{F}_t^{-1}\big[r^n(\lambda)\widehat{\mu}(\lambda)\big](t). \tag{19}$$

Proof. This assertion obviously follows from (14) since $\Re r(\lambda) \leq 0$.

DEFINITION 2. *For any $t \in \mathbb{R}$ and any $x < 0$ let*

$$\widetilde{J}(t, x; \mu) \equiv 3 \int_t^{+\infty} \frac{1}{\tau - t} A''\left(\frac{-x}{(\tau - t)^{1/3}}\right) \mu(\tau)\, d\tau. \tag{20}$$

It is obvious, that

$$\widetilde{J}(t, x; \mu) = J(-t, -x; \widetilde{\mu}), \qquad \text{where} \quad \widetilde{\mu}(t) \equiv \mu(-t), \tag{21}$$

and therefore, the properties established in the previous lemmas for the function J are also valid for the function $\widetilde{J}$ with natural substitution of $\mathbb{R}_+$ by $\mathbb{R}_-$.

Now we turn to the problem in Q_T. The definition of a generalized solution of the problem (1)–(3) can be found in [4]. For integer $k \geq 0$ define special functional space for solutions

$$X_k(Q_T) = \{u(t,x) : D_t^m u \in C([0,T]; H^{3(k-m)}(0,1)) \cap L_2(0,T; H^{3(k-m)+1}(0,1)) \\ \text{for } 0 \leq m \leq k\}$$

and for right parts of the equation

$$M_k(Q_T) = \{f(t,x) : D_t^m f \in C([0,T]; H^{3(k-m-1)}(0,1)) \cap L_2(0,T; H^{3(k-m-1)+1}(0,1)) \\ \text{for } 0 \leq m \leq k-1, \quad D_t^k f \in L_1(0,T; L_2(0,1))\}.$$

In this article we consider only the cases $k = 0$ and $k = 1$.

THEOREM 1. *Let $k = 0$ or $k = 1$, the function $g \in C^{k+1}(\mathbb{R})$ and satisfies the inequality (5). Assume, that $u_0 \in H^{3k}(0,1)$, $u_1, u_2 \in (H^{k+1/3} \cap W_1^{1/3})(0,T)$, $u_3 \in H^k(0,T)$, $f \in M_k(Q_T)$ for some $T > 0$ and if $k = 1$, then additionally $u_0(0) = u_1(0)$, $u_0(1) = u_2(0)$, $u_0'(1) = u_3(0)$. Then there exists a unique solution $u(t,x)$ of the problem (1)–(3) in the space $X_k(Q_T)$. The mapping $(u_0, u_1, u_2, u_3, f) \mapsto u$ is Lipschitz continuous on any ball in the norm of the mapping $H^{3k}(0,1) \times \left((H^{k+1/3} \cap W_1^{1/3})(0,T)\right)^2 \times H^k(0,T) \times M_k(Q_T) \to X_k(Q_T)$.*

Proof. The proof consists of three steps.

Step 1. The linear problem. Consider in Q_T the problem

$$v_t + v_{xxx} = f(t,x), \tag{22}$$

$$v\big|_{t=0} = u_0, \quad v\big|_{x=0} = u_1, \quad v\big|_{x=1} = u_2, \quad v_x\big|_{x=1} = u_3. \tag{23}$$

Define an auxiliary function

$$\psi(t, x; u_1, u_2) \equiv J(t, x; u_1)\eta(1-x) + \widetilde{J}(t, x-1; u_2)\eta(x). \tag{24}$$

Let

$$V(t,x) \equiv u(t,x) - \psi(t, x; u_1, u_2), \quad F(t,x) \equiv f(t,x) - (\psi_t + \psi_{xxx}),$$
$$U_0(x) \equiv u_0(x) - \psi(0,x), \quad U_3(t) \equiv u_3(t) - \widetilde{J}_x(t, 0-0; u_2).$$

Then the function v is a solution of the problem (22)–(23) if and only if the function V is a solution of the problem

$$V_t + V_{xxx} = F(t,x), \tag{25}$$
$$V\big|_{t=0} = U_0, \quad V\big|_{x=0} = V\big|_{x=1} = 0, \quad V_x\big|_{x=1} = U_3.$$

Note, that if the conditions of the considered theorem are satisfied for $k = 0$, then in virtue of Lemmas 1-5 $F \in L_1(0,T; L_2(0,1))$, $U_0 \in L_2(0,1)$, $U_3 \in L_2(0,T)$, $\psi \in X_0(Q_T)$ with corresponding estimates by norms of u_0, u_1, u_2, u_3 and f.

After that, using the same arguments as in [4, Lemma 2] one can prove, that there exists a unique solution $v(t,x)$ of the problem (22)–(23) in the spaces $X_k(Q_T)$ and for any $t_0 \in (0,T]$

$$\|v\|_{X_k(Q_{t_0})} \le c(T)\big(\|u_0\|_{H^{3k}(0,1)} + \|u_1\|_{H^{k+1/3}(0,T)} + \|u_2\|_{H^{k+1/3}(0,T)} + \\ + \|u_3\|_{H^k(0,T)} + \|f\|_{M_k(Q_{t_0})}\big). \tag{26}$$

Morover, for every $t \in [0,T]$ and $\rho(x) \equiv 1$ or $\rho(x) \equiv 1 + x$

$$\int_0^1 V^2(t,x)\rho(x)\,dx + 3\int_0^t\!\int_0^1 V_x^2\rho'\,dxd\tau \le 4\int_0^1 u_0^2\,dx + 2\int_0^t\!\int_0^1 fV\rho\,dxd\tau + \\ + c(T)\int_0^t\!\int_0^1 V^2\rho\,dxd\tau + c(T)\left(\|u_1\|^2_{H^{1/3}(0,T)} + \|u_2\|^2_{H^{1/3}(0,T)} + \|u_3\|^2_{L_2(0,T)}\right) \tag{27}$$

(for solutions from $X_1(Q_T)$ this inequality can be obviously obtained via multiplying (25) by $V(t,x)\rho(x)$ and integrating and then for solutions from $X_0(Q_T)$ via conclusion) and if $k = 1$, then the function v_t is a solution in Q_T of a problem of the (1)–(3) type, where u_0, u_1, u_2, u_3, f are substituted by $f|_{t=0} - u_0'''$, u_1', u_2', u_3', f_t.

Step2. Local well-posedness. For any $t_0 \in (0,T]$ define classes of functions $Z_0(Q_{t_0}) = X_0(Q_{t_0})$,

$$Z_1(Q_{t_0}) = \{u \in X_1(Q_{t_0}) : u\big|_{t=0} = u_0\}$$

and define on these classes a map Λ in such a way: $v = \Lambda u$ for $u \in Z_k(Q_{t_0})$, if $v \in Z_k(Q_{t_0})$ and solves in Q_{t_0} a linear initial boundary value problem for the equation

$$v_t + v_{xxx} = f - g(u)u_x \tag{28}$$

with initial and boundary conditions (23).

If $k = 0$, then

$$\|g(u)u_x\|_{L_1(0,t_0;L_2(0,1))} \le c\int_0^{t_0}\big(1 + \|u\|_{C[0,1]}\big)\|u_x\|_{L_2(0,1)}\,dt \le \\ \le c_1\int_0^{t_0}\big(\|u_x\|^{3/2}_{L_2(0,1)}\|u\|^{1/2}_{L_2(0,1)} + \|u\|^2_{L_2(0,1)} + \|u_x\|_{L_2(0,1)}\big)\,dt \le c(T)t_0^{1/4}\big(1 + \|u\|^2_{X_0(Q_{t_0})}\big). \tag{29}$$

If $k=1$, then

$$\begin{gathered}\|(g(u)u_x)_t\|_{L_1(0,t_0;L_2(0,1))} \leq \\ \leq c\|u_t\|_{L_1(0,t_0;C[0,1])}\|u_x\|_{C([0,t_0];L_2(0,1))} + c\|u_{tx}\|_{L_1(0,t_0;L_2(0,1))}\bigl(1+\|u\|_{C(\overline{Q}_{t_0})}\bigr) \leq \\ \leq c_1 t_0^{1/2}\|u_t\|_{L_2(0,t_0;H^1(0,1))}\bigl(1+\|u_0\|_{H^1(0,1)}+\|u_t\|_{L_1(0,t_0;H^1(0,1))}\bigr) \leq \\ \leq c(T)\bigl(1+t_0\|u\|^2_{X_1(Q_{t_0})}\bigr),\end{gathered} \tag{30}$$

$$\begin{gathered}\|(g(u)u_x)_x\|_{L_2(Q_{t_0})} \leq \\ \leq c\|u_x\|_{L_2(0,t_0;C[0,1])}\|u_x\|_{C([0,t_0];L_2(0,1))} + c\|u_{xx}\|_{L_2(Q_{t_0})}\bigl(1+\|u\|_{C(\overline{Q}_{t_0})}\bigr) \leq \\ \leq c_1 t_0^{1/2}\|u\|_{C([0,t_0];H^3(0,1))}\bigl(1+\|u_0\|_{H^1(0,1)}+\|u_t\|_{L_1(0,t_0;H^1(0,1))}\bigr) \leq \\ \leq c(T)\bigl(1+t_0\|u\|^2_{X_1(Q_{t_0})}\bigr).\end{gathered} \tag{31}$$

Therefore, the desired solution $v(t,x)$ of the problem (28),(23) exists and according to (26) and (29)–(31)

$$\|v\|_{X_k(Q_{t_0})} \leq \widetilde{c}\left(1+t_0^{1/4}\|u\|^2_{X_k(Q_{t_0})}\right).$$

Choosing $B \geq 2\widetilde{c}$ and $t_0 \leq (2\widetilde{c}B)^{-4}$ we find, that Λ transforms a ball $Z_{k,B}(Q_{t_0}) = \{u \in Z_k(Q_{t_0}) : \|u\|_{X_k(Q_{t_0})} \leq B\}$ into itself.

Next, we consider two functions u and $\widetilde{u}$ from $Z_{k,B}(Q_{t_0})$. Let $v=\Lambda u$, $\widetilde{v}=\Lambda\widetilde{u}$, $w=v-\widetilde{v}$, $\widetilde{w}=u-\widetilde{u}$. Then w solves in Q_{t_0} a problem for the equation

$$w_t + w_{xxx} = -\bigl(g(u)u_x - g(\widetilde{u})\widetilde{u}_x\bigr)$$

with zero initial and boundary conditions. By analogy with (29)–(31) one can derive, that

$$\|g(u)u_x - g(\widetilde{u})\widetilde{u}_x\|_{M_k(Q_{t_0})} \leq c(B)t_0^{1/4}\|\widetilde{w}\|_{X_k(Q_{t_0})},$$

whence with the use of (26) obtain, that

$$\|w\|_{X_k(Q_{t_0})} \leq c(B)t_0^{1/4}\|\widetilde{w}\|_{X_k(Q_{t_0})}.$$

Thus, for sufficient small t_0 the map Λ is a contraction on $Z_{k,B}(Q_{t_0})$ and, therefore, in Q_{t_0} there exists a unique solution $u \in X_k(Q_{t_0})$ of the problem (1)–(3).

The continuous dependence result can be proved by a similar argument.

Step 3. Nonlocal apriori estimates. Let $u \in X_0(Q_{t_0})$ be a solution of the problem (1)–(3) in Q_{t_0} for some $t_0 \in (0,T]$. We establish the following estimate:

$$\begin{gathered}\|u\|_{X_0(Q_{t_0})} \leq c\bigl(T, \|u_0\|_{L_2(0,1)}, \|u_1\|_{(H^{1/3}\cap W_1^{1/3})(0,T)}, \|u_2\|_{(H^{1/3}\cap W_1^{1/3})(0,T)}, \\ \|u_3\|_{L_2(0,T)}, \|f\|_{L_1(0,T;L_2(0,1))}\bigr).\end{gathered} \tag{32}$$

To this end first of all choosing in (27) $\rho \equiv 1$ we derive for

$$U(t,x) \equiv u(t,x) - \psi(t,x;u_1,u_2)$$

(ψ is defined by the formula (24)) the following inequality

$$\int_0^1 U^2(t,x)\,dx \le P_0 + c\int_0^t\int_0^1 U^2\,dxd\tau + 2\int_0^t\int_0^1 (f-g(u)u_x)U\,dxd\tau, \tag{33}$$

where here and further in the proof we denote by P_0 arbitrary constants depending on the same values as the constant in the right part of (32). Note, that

$$g(u)u_xU = D_x\Big(\int_0^U g(\theta+\psi)\theta\,d\theta\Big) + \psi_x\int_0^U g(\theta+\psi)\,d\theta \tag{34}$$

and, therefore,

$$\Big|\int_0^t\int_0^1 g(u)u_xU\,dxd\tau\Big| = \Big|\int_0^t\int_0^1 \psi_x\int_0^U g(\theta+\psi)\,d\theta dxd\tau\Big| \le$$

$$\le c\int_0^t \|\psi_x\|_{C[0,1]}d\tau\Big[1+\sup_{\tau\in[0,t]}\int_0^1(U^2+\psi^2)dx\Big].$$

Thus, using the inequalities (13), (17) and the formula (21), we derive from (33) the estimate

$$\|u\|_{C([0,t_0];L_2(0,1))} \le P_0. \tag{35}$$

Then, again using the inequality (27) (for $V\equiv U$) and choosing $\rho\equiv 1+x$ we derive (taking into account (35)), that

$$3\int_0^{t_0}\int_0^1 U_x^2\,dxdt \le P_0 - 2\int_0^{t_0}\int_0^1 (1+x)g(u)u_xU\,dxdt.$$

Again using (34) we find, that

$$\Big|\int_0^{t_0}\int_0^1(1+x)g(u)u_xU\,dxdt\Big| \le$$

$$\le c\int_0^{t_0}\Big(\|U\|_{C[0,1]}+\|\psi_x\|_{C[0,1]}+1\Big)\Big[1+\int_0^1(U^2+\psi^2)dx\Big]dt \le \Big(\int_0^{t_0}\int_0^1 U_x^2\,dxdt\Big)^{1/4} + P_0,$$

and derive (also with the use of the estimate (18)), that

$$\|u_x\|_{L_2(Q_{t_0})} \le P_0. \tag{36}$$

The estimates (35) and (36) provide the inequality (32).

Now for some $t_0 \in (0,T]$ consider the solution $u(t,x)$ of the problem (1)–(3) in $X_1(Q_{t_0})$ and prove that

$$\|u\|_{X_1(Q_{t_0})} \le c\big(T, \|u_0\|_{L_2(0,1)}, \|u_1\|_{(H^{1/3}\cap W_1^{1/3})(0,T)}, \|u_2\|_{(H^{1/3}\cap W_1^{1/3})(0,T)},$$
$$\|u_3\|_{L_2(0,T)}, \|f\|_{L_1(0,T;L_2(0,1))}\big) \times \tag{37}$$
$$\times \big(\|u_0\|_{H^3(0,1)} + \|u_1\|_{H^{4/3}(0,T)} + \|u_2\|_{H^{4/3}(0,T)} + \|u_3\|_{H^1(0,T)} + \|f\|_{M_1(Q_T)}\big).$$

To this end we first of all consider the function $v \equiv u_t$, which solves in Q_{t_0} the linear problem

$$v_t + v_{xxx} = f_t - (g(u)u_x)_t,$$

$$v\big|_{t=0} = f\big|_{t=0} - g(u_0)u_0' - u_0''', \quad v\big|_{x=0} = u_1', \quad v\big|_{x=1} = u_2', \quad v_x\big|_{x=1} = u_3'.$$

Let $\psi_1 \equiv \psi(t,x;u_1',u_2')$. We put $U_1(t,x) \equiv u_t(t,x) - \psi_1(t,x)$. Then using for this problem the inequality (27) with $\rho(x) \equiv 1+x$ we find, that

$$\int_0^1 (1+x)U_1^2(t,x)\,dx + 3\int_0^t\!\!\int_0^1 U_{1x}^2\,dxd\tau \le$$
$$\le P_1^2 + c\int_0^t\!\!\int_0^1 (1+x)U_1^2\,dxd\tau + 2\int_0^t\!\!\int_0^1 (1+x)\big(f_\tau - (g(u)u_x)_\tau\big)U_1\,dxd\tau, \tag{38}$$

where here and further in the proof we denote by P_1 arbitrary constants of the same structure as the right part of (37). For any $\delta > 0$

$$\Big|\int_0^t\!\!\int_0^1 (1+x)\big(g'(u)u_\tau u_x + g(u)u_{x\tau}\big)U_1\,dxd\tau\Big| \le c\int_0^t \Big[\|u_x\|_{L_2(0,1)}\|U_1 +$$
$$+\psi_1\|_{L_2(0,1)} + \big(1+\|u\|_{L_2(0,1)}\big)\|U_{1x}+\psi_{1x}\|_{L_2(0,1)}\Big]\|U_1\|_{C[0,1]}\,d\tau \le$$
$$\le \delta\int_0^t\!\!\int_0^1 U_{1x}^2\,dxd\tau + c(\delta)\int_0^t \|u_x\|^{4/3}_{L_2(0,1)}\int_0^1 (1+x)U_1^2\,dxd\tau +$$
$$+c(\delta)\left(1+\|u_x\|^{4/3}_{L_2(Q_{t_0})}\right)\|\psi_1\|^2_{C([0,T];L_2(0,1))} + \tag{39}$$
$$+c(\delta)\left(1+\|u\|^4_{C([0,t_0];L_2(0,1))}\right)\int_0^t\!\!\int_0^1 (1+x)U_1^2\,dxd\tau + c(\delta)\|\psi_{1x}\|^2_{L_2(Q_T)}.$$

Using the inequality (32) and the corresponding estimates for the function ψ_1 we derive from (38) and (39), that

$$\|u_t\|_{X_0(Q_{t_0})} \le P_1. \tag{40}$$

Next, with the use of (40) and the inequality

$$\|g(u)u_x\|_{C([0,t_0];H^{-1}(0,1))} \le c\|(1+|u|)u\|_{C([0,t_0];L_2(0,1))} \le$$
$$\le c_1\big[1+\|u\|_{C([0,t_0];L_2(0,1))}\big]\|u\|_{C(\overline{Q}_{t_0})} \le c(\|u\|_{X_0(Q_{t_0})})(\|u_0\|_{H^1(0,1)} + \|u_t\|_{X_0(Q_{t_0})})$$

we find, expressing u_{xxx} from the equation (1) itself, that

$$\|u_{xx}\|_{C([0,t_0];L_2(0,1))} \le c\big[\|u_{xxx}\|_{C([0,t_0];H^{-1}(0,1))} + \|u\|_{C([0,t_0];L_2(0,1))}\big] \le P_1. \tag{41}$$

Again using the equation (1) and taking into account (41) we derive the estimate

$$\|u_{xxx}\|_{C([0,t_0];L_2(0,1))} \le P_1, \tag{42}$$

since

$$\|g(u)u_x\|_{C([0,t_0];L_2(0,1))} \le c\big[1 + \|u\|_{C([0,t_0];L_2(0,1))}\big]\|u_x\|_{C(\overline{Q}_{t_0})}.$$

Finally, differentiating the equation (1) with respect to x and expressing u_{xxxx} we find, that

$$\|u_{xxxx}\|_{L_2(Q_{t_0})} \le P_1, \tag{43}$$

since

$$\|g'(u)u_x^2 + g(u)u_{xx}\|_{L_2(Q_{t_0})} \le c\|u_x\|_{L_2(Q_{t_0})}\|u_x\|_{C(\overline{Q}_{t_0})}+$$

$$+c\big[1 + \|u\|_{L_2(0,t_0;C[0,1])}\big]\|u_{xx}\|_{C([0,t_0];L_2(0,1))} \le c(\|u\|_{X_0(Q_{t_0})})\|u\|_{C([0,t_0];H^3(0,1))} \le P_1.$$

The estimate (37) follows from (40), (42) and (43). The theorem is proved.

REMARK 1. *It is obvious, that* $H^{1/3+\varepsilon}(0,T) \subset (H^{1/3} \cap W_1^{1/3})(0,T)$ *for any* $\varepsilon > 0$, $H^{4/3}(0,T) = (H^{4/3} \cap W_1^{1/3})(0,T)$.

REMARK 2. *With the use of the properties of the boundary potential J, obtained in Lemmas 1–5, nonlocal well-posedness of the problem (1)–(3) can be established in the spaces $X_k(Q_T)$ for natural $k \ge 2$, if $u_0 \in H^{3k}(0,1)$, $u_1, u_2 \in H^{k+1/3}(0,T)$, $u_3 \in H^k(0,T)$, $f \in M_k(Q_T)$, the function g is sufficiently smooth and certain compatibility conditions are satisfied.*

Now we consider the problem in Π_T^-. The definition of a generalized solution of the problem (1),(2),(4) can be found in [5]. Let

$$\lambda_0^-(u;T) = \sup_{m\ge 0}\Big(\int_0^T\int_{-m-1}^{-m} u^2(t,x)\,dxdt\Big)^{1/2},$$

$$\lambda_2^-(u;T) = \Big(\sum_{m=0}^{+\infty}\sup_{(t,x)\in[0,T]\times[-m-1,-m]} u^2(t,x)\Big)^{1/2}$$

and define special functional spaces

$$X^w(\Pi_T^-) = \{u(t,x) : u \in C_{b,w}([0,T];L_2(\mathbb{R}_-)), \lambda_0^-(u_x;T) < \infty\},$$

$$Y(\Pi_T^-) = \{u(t,x) : u \in C([0,T];H^2(\mathbb{R}_-)), \lambda_0^-(u_{xxx};T) + \lambda_2^-(u;T) < \infty\}.$$

First of all we study solutions from the class $X^w(\Pi_T^-)$.

THEOREM 2. *Let the function $g \in C^1(\mathbb{R})$ and satisfies the inequality (5). Assume, that $u_0 \in L_2(\mathbb{R}_-)$, $u_2 \in (H^{1/3} \cap W_1^{1/3})(0,T)$, $u_3 \in L_2(0,T)$, $f \in L_1(0,T;L_2(\mathbb{R}_-))$ for some $T > 0$. Then there exists a unique solution $u(t,x)$ of the problem (1),(2),(4) in the space $X^w(\Pi_T^-)$.*

If a function $\widetilde{u}(t,x) \in X^w(\Pi_T^-)$ is a solution of the problem of the (1),(2),(4) type, where u_0, u_2, u_3, f are substituted by $\widetilde{u}_0$, $\widetilde{u}_2$, $\widetilde{u}_3$, $\widetilde{f}$ from the same spaces, then for every $\beta > 0$

$$\begin{aligned} \big\|(u-\widetilde{u})e^{\beta x}\big\|_{C([0,T];L_2(\mathbb{R}_-))} + \sqrt{\beta}\big\|(u-\widetilde{u})_x e^{\beta x}\big\|_{L_2(\Pi_T^-)} \le c\Big[\big\|(u_0-\widetilde{u}_0)e^{\beta x}\big\|_{L_2(\mathbb{R}_-)} + \\ + \big\|u_2-\widetilde{u}_2\big\|_{H^{1/3}(0,T)} + \big\|u_3-\widetilde{u}_3\big\|_{L_2(0,T)} + \big\|(f-\widetilde{f})e^{\beta x}\big\|_{L_1(0,T;L_2(\mathbb{R}_-))}\Big], \end{aligned} \tag{44}$$

where the constant c depends on β, T, properties of the function g and norms of the functions u and $\widetilde{u}$ in the space $L_\infty(0,T;L_2(\mathbb{R}_-))$.

Proof. The exact analogue of this result has been previously established in [5] under the condition $u_2 \in W_1^{5/6+\varepsilon}(0,T)$ for some $\varepsilon > 0$. The proof of the present theorem is carried out quite similar to [5] with the use of the function $\widetilde{J}$ instead of J_0. That is why we point out here only the main ideas of the proof.

Generalized solutions of the considered problem are constructed as limits while $\delta \to +0$ of solutions of regularized equations

$$u_t + u_{xxx} - 3\delta u_{xx} + g(u)u_x = f(t,x), \quad \delta > 0 \tag{45}$$

(g, f, u_0, u_2, u_3 are also regularized). We prove for (regular) solutions of the problem (45),(2),(4) in Π_T^- an apriori estimate independent on δ, which is crucial for the passage to the limit:

$$\|u\|_{X^w(\Pi_T^-)} \le c(T, \|u_0\|_{L_2(\mathbb{R}_-)}, \|u_2\|_{(H^{1/3}\cap W_1^{1/3})(0,T)}, \|u_3\|_{L_2(0,T)}, \|f\|_{L_1(0,T;L_2(\mathbb{R}_-))}). \tag{46}$$

To this end we define an auxiliary function

$$\psi(t,x;u_2) \equiv \widetilde{J}(t,x;u_2)e^{\delta x}$$

and for the function $U(t,x) \equiv u(t,x) - \psi(t,x;u_2)$ consider the problem

$$\begin{gathered} U_t + U_{xxx} - 3\delta U_{xx} + g(U+\psi)(U+\psi)_x = F(t,x), \\ U\big|_{t=0} = U_0, \quad U\big|_{x=0} = 0, \quad U_x\big|_{x=0} = U_3, \end{gathered} \tag{47}$$

where $F \equiv f - (\psi_t + \psi_{xxx} - 3\delta\psi_{xx})$, $U_0(x) \equiv u_0(x) - \psi(0,x)$, $U_3(t) \equiv u_3(t) - \psi_x(t,-0)$. It is easy to see, that

$$\psi_t + \psi_{xxx} - 3\delta\psi_{xx} = -3\delta^2\widetilde{J}_x e^{\delta x} - 2\delta^3\widetilde{J}e^{\delta x}$$

and, therefore,

$$\|\psi_t + \psi_{xxx} - 3\delta\psi_{xx}\|_{L_1(0,T;L_2(\mathbb{R}_-))} \le c(T)\|u_2\|_{H^{1/3}(0,T)}.$$

Let $\rho(x) \in C^3(\mathbb{R})$ be a certain positive, nondecreasing function such, that $|\rho^{(j)}(x)| \le c\rho(x)$ for $x \le 0$ and $j = 1,2,3$. Then multiplying (47) by $2U(t,x)\rho(x)$ and integrating we find, that

$$\begin{aligned} \int_{\mathbb{R}_-} U^2(t,x)\rho\,dx + 3\int_0^t\int_{\mathbb{R}_-} U_x^2\rho'\,dxd\tau + 6\delta\int_0^t\int_{\mathbb{R}_-} U_x^2\rho\,dxd\tau \le \int_{\mathbb{R}_-} U_0^2\rho\,dx + \\ +2\int_0^t\int_{\mathbb{R}_-}(F - g(u)u_x)U\rho\,dxd\tau + c\int_0^t\int_{\mathbb{R}_-} U^2\rho\,dxd\tau + \rho(0)\int_0^t U_3^2\,d\tau. \end{aligned} \tag{48}$$

The equality (34) yields, that

$$\int_{\mathbb{R}_-} g(u)u_x U\rho\,dx = -\int_{\mathbb{R}_-}\int_0^U g(\theta+\psi)\theta d\theta\rho'\,dx + \int_{\mathbb{R}_-}\psi_x\int_0^U g(\theta+\psi)d\theta\rho\,dx. \tag{49}$$

At first, choosing in (48) and (49) $\rho\equiv 1$ we find similarly to (35), that

$$\|u\|_{C([0,T];L_2(\mathbb{R}_-))} \le P, \tag{50}$$

where here and further in the proof we denote by P arbitrary constants, depending on the same values, as the constant in the right part of (46). Next, choosing

$$\rho(x) = \rho_m(x) \equiv 1 + \eta((x+m+2)/3)$$

for every integer $m\ge 0$ we find similarly to (36), that

$$\lambda_0^-(u_x;T) \le P. \tag{51}$$

The estimate (46) follows from (50) and (51). The rest part of the proof of existence of generalized solutions is just the same as in [5, Theorem 1].

For the proof of the inequality (44) one has to write down the analogue of the inequality (48) for the function $V(t,x)\equiv u(t,x)-\widetilde{u}(t,x)-\widetilde{J}(t,x;u_2-\widetilde{u}_2)$ (here, of course, $\delta=0$) and $\rho(x)\equiv\exp(2\beta x)$ and estimate in a proper way the nonlinear term (for more details see [5, Theorem 2] or in the case of the KdV equation itself [3, Theorem 4.3]). The proof is completed.

More smooth nonlocal solutions of the problem in Π_T^- are constructed only in the case of the KdV equation itself.

THEOREM 3. *Let $u_0\in H^2(\mathbb{R}_-)$, $u_2\in H^1(0,T)$, $u_3\in H^{2/3}(0,T)$, $f\in L_2(0,T;H^2(\mathbb{R}_-))$ for some $T>0$ and $u_0(0)=u_2(0)$, $u_0'(0)=u_3(0)$. Then there exists a unique solution $u(t,x)$ of the problem (6),(2),(4) in the space $Y(\Pi_T^-)$. The mapping $(u_0,u_2,u_3,f)\mapsto u$ is Lipschitz continuous on any ball in the norm of the mapping $H^2(\mathbb{R}_-)\times H^1(0,T)\times H^{2/3}(0,T)\times L_2(0,T;H^2(\mathbb{R}_-))\to Y(\Pi_T^-)$.*

Proof. The exact analogue of this result has been previously established in [3] under the condition $u_3\in W^1_{4/3}(0,T)$. As for the previous theorem we point out here only the main ideas of the proof.

At first consider in Π_T^- the linear problem

$$v_t + v_{xxx} + av_x = f(t,x), \tag{52}$$

$$v\big|_{t=0} = u_0, \quad v\big|_{x=0} = u_2, \quad v_x\big|_{x=0} = u_3 \tag{53}$$

and prove for a solution $v(t,x)\in Y(\Pi_T^-)$ the following three inequalities: for any $t_0\in(0,T]$

$$\|v\|_{Y(\Pi_{t_0}^-)} \le c(a,T)\big(\|u_0\|_{H^2(\mathbb{R}_-)} + \|u_2\|_{H^1(0,T)} + \|u_3\|_{H^{2/3}(0,T)} + \\ + \|f\|_{L_1(0,t_0;H^2(\mathbb{R}_-))} + \|f(\cdot,0)\|_{L_2(0,t_0)} + \|f_x(\cdot,0)\|_{L_{4/3}(0,t_0)}\big); \tag{54}$$

if

$$\psi(t,x) = \psi(t,x;u_3) \equiv \eta(x+1)\int\limits_0^x \widetilde{J}(t,y;u_3)\,dy, \quad V(t,x) \equiv v(t,x) - \psi(t,x),$$

then for $t \in (0,T]$ and a function $\rho(x)$, which satisfies the same conditions as in the inequality (48) and, in addition, $\rho(x) \geq 1$ for $x \leq 0$, $\rho'(x) > 0$ for $x \in [-1,0]$,

$$\begin{gathered}\int\limits_{\mathbb{R}_-} V_{xx}^2(t,x)\rho(x)\,dx + 2\int\limits_0^t\int\limits_{\mathbb{R}_-} V_{xxx}^2\rho'\,dxd\tau \leq 2\int\limits_{\mathbb{R}_-}(u_0'')^2\rho\,dx + c\int\limits_0^t\int\limits_{\mathbb{R}_-} V_{xx}^2\rho\,dxd\tau + \\ +2\int\limits_0^t\int\limits_{\mathbb{R}_-} f_{xx}V_{xx}\rho\,dxd\tau + c\|u_2\|^2_{H^1(0,T)} + c\|u_3\|^2_{H^{2/3}(0,T)} + c\int\limits_0^t f^2(\tau,0)\,d\tau + \\ +c\int\limits_0^t |f_x(\tau,0)|^{4/3}\Big(\int\limits_{\mathbb{R}_-} V_{xx}^2\rho\,dx\Big)^{1/3}\,d\tau,\end{gathered} \tag{55}$$

where the constant c depends on a, T and the properties of the function ρ;

$$\begin{gathered}\tfrac{5}{6}\int\limits_{\mathbb{R}_-} V^2(t,x)V_{xx}(t,x)\rho(x)\,dx - 5\int\limits_0^t\int\limits_{\mathbb{R}_-} V_xV_{xx}^2\rho\,dxd\tau \leq \int\limits_0^t\int\limits_{\mathbb{R}_-} V_{xxx}^2\rho'\,dxd\tau + \\ +c\int\limits_0^t\Big(\sup_{x\in\mathbb{R}_-} f^2(\tau,x) + \int\limits_{\mathbb{R}_-} f^2(\tau,x)\,dx\Big)d\tau + c\int\limits_0^t\int\limits_{\mathbb{R}_-} V_{xx}^2\rho\,dxd\tau + c,\end{gathered} \tag{56}$$

where in this inequality the constant c depends also on $\|u_0\|_{H^2(\mathbb{R}_-)}$, $\|u_2\|_{H^1(0,T)}$, $\|u_3\|_{H^{2/3}(0,T)}$ and $\|v\|_{C([0,T];L_2(\mathbb{R}_-))}$.

Note, that the function $V(t,x)$ is a solution of the problem

$$\begin{gathered}V_t + V_{xxx} + aV_x = f - (\psi_t + \psi_{xxx} + a\psi_x), \\ V\big|_{t=0} = u_0 - \psi\big|_{t=0}, \quad V\big|_{x=0} = u_2, \quad V_x\big|_{x=0} = 0.\end{gathered} \tag{57}$$

Multiplying (57) by $2V_{xxxx}(t,x)\rho(x)$ and integrating, using the evident interpolation inequality (following from the properties of the function ρ)

$$\big|V_{xx}\big|_{x=0}\big| \leq c\Big[\Big(\int\limits_{\mathbb{R}_-} V_{xxx}^2\rho'\,dx\Big)^{1/4}\Big(\int\limits_{\mathbb{R}_-} V_{xx}^2\rho\,dx\Big)^{1/4} + \Big(\int\limits_{\mathbb{R}_-} V_{xx}^2\rho\,dx\Big)^{1/2}\Big], \tag{58}$$

we derive the inequality (55). Choosing in (55)

$$\rho(x) = \rho_m(x) \equiv 1 + \eta((x+2)/3) + \eta((x+m+2)/3)$$

we find, that $\|u_{xx}\|_{C([0,t_0];L_2(\mathbb{R}_-)))}$ and $\lambda_0^-(u_{xxx};t_0)$ are estimated by the left part of (54). Estimating $\lambda_2^-(u;t_0)$ just as in [3, Corollary 3.3], we finish the proof of (54). Finally, multiplication of (57) by $\frac{5}{3}\big(V_x^2(t,x) + 2V(t,x)V_{xx}(t,x)\big)\rho(x)$ and integration provides the inequality (56) (again with the use of (58), for more details see [3, Lemma 3.3]).

Next, local well-posedness for the KdV equation is established via the contraction principle on the basis of the estimate (54) (for more details see [3, Theorem 4.1], similar arguments are carried out in Theorem 1 of the present paper).

In order to establish global well-posedness we prove for a solution $u(t,x) \in Y(\Pi_T^-)$ the following a priori estimate:

$$\|u\|_{Y(\Pi_T^-)} \leq c(a,T,\|u_0\|_{H^2(\mathbb{R}_-)},\|u_2\|_{H^1(0,T)},\|u_3\|_{H^{2/3}(0,T)},\|f\|_{L_2(0,T;H^2(\mathbb{R}_-))}). \tag{59}$$

To this end we write down the inequalities (55) and (56) in the case of the equation (6) (here f must be substituted by $f - uu_x$) for $\rho(x) \equiv \rho_m(x)$. Then summation of these inequalities provides the estimate

$$\|u\|_{C([0,T];H^2(\mathbb{R}_-))} + \lambda_0^-(u_{xxx};T) \leq P, \tag{60}$$

where here and further in the proof we denote by P arbitrary constants, depending on the same values as the constant in the right part of (59). Next, since for $t_0 \in (0,T]$

$$\|uu_x\|_{L_1(0,t_0;H^2(\mathbb{R}_-))} \leq ct_0^{1/2}\big(\|u\|^2_{C([0,t_0];H^2(\mathbb{R}_-))} + \lambda_2^-(u;t_0)\lambda_0^-(u_{xxx};t_0)\big),$$

$$\|u_{xx}(\cdot,0)\|_{L_{4/3}(0,t_0)} \leq ct_0^{1/2}\big(\lambda_0^-(u_{xxx};t_0) + \|u\|_{C([0,t_0];H^2(\mathbb{R}_-))}\big),$$

we obtain from (54) and (60), that

$$\|u\|_{Y(\Pi^-_{t_0})} \leq P\left[1 + t_0^{1/2}\|u\|_{Y(\Pi^-_{t_0})}\right],$$

and, therefore, establish (59). The theorem is proved.

REMARK 3. *In more smooth classes nonlocal well–posedness of the problem (6),(2),(4) in Π_T^- can be established for natural $k \geq 2$, if $u_0 \in H^{3k}(\mathbb{R}_-)$, $u_2 \in H^{k+1/3}(0,T)$, $u_3 \in H^k(0,T)$ or $u_0 \in H^{3k+2}(\mathbb{R}_-)$, $u_2 \in H^{k+1}(0,T)$, $u_3 \in H^{k+2/3}(0,T)$, the function f is suficiently smooth and certain compatibility conditions are satisfied (see [3, Theorems 5.1 and 5.2]).*

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